LONG-TIME ASYMPTOTIC FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH STEP-LIKE INITIAL VALUE

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Abstract. We consider the Cauchy problem for the Gerdjikov-Ivanov(GI) type of the derivative nonlinear Schrödinger (DNLS) equation:

$$iq_t + q_{xx} - iq^2\bar{q}_x + \frac{1}{2}|q|^4 q = 0.$$

with steplike initial data: $q(x,0) = 0$ for $x \leq 0$ and $q(x,0) = Ae^{-2iBx}$ for $x > 0$,where $A > 0$ and $B \in \mathbb{R}$ are constants.The paper aims at studying the long-time asymptotics of the solution to this problem.We show that there are four regions in the half-plane $-\infty < x < \infty, t > 0$,where the asymptotics has qualitatively different forms:a slowly decaying self-similar wave of Zakharov-Manakov type for $x > -4tB$, a plane wave region:$x < -4t(B + \sqrt{2A^2(B+\frac{A^2}{4})})$, an elliptic region:$-4t(B+\sqrt{2A^2(B+\frac{A^2}{4})}) < x < -4tB$. The main tool is the asymptotic analysis of an associated matrix Riemann-Hilbert problem.

1. Introduction

The classical, mathematical model for non-linear pulse propagation in the picosecond time scale in the anomalous dispersion regime in an isotropic, homogeneous, lossless, non-amplifying, polarization-preserving single-mode optical fibre is the non-linear Schrödinger(NLS) equation [2]. However, in the subpicosecond-femtosecond time scale, experiments and theories on the propagation of high-power ultrashort pulses in long monomode optical fibres have shown that the NLS equation is

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no longer valid and that additional non-linear terms (dispersive and dissipative) and higher-order linear dispersion should be taken into account, you can see [36] and the references therein. In this case, subpicosecond-femtosecond pulse propagation is described (in dimensionless and normalized form) by the following non-linear evolution equation (NLEE)

$$iu_\xi + \frac{1}{2}u_{\tau\tau} + |u|^2u + is(|u|^2u)_\tau = -i\tilde{\Gamma}u + i\tilde{\delta}u_{\tau\tau\tau} + \frac{\tau_n}{\tau_0}u(|u|^2)_\tau, \quad (1.1)$$

where u is the slowly varying amplitude of the complex field envelope, ξ is the propagation distance along the fibre length, τ is the time measured in a frame of reference moving with the pulse at the group velocity (the retarded frame), $s(>0)$ governs the effects due to the intensity dependence of the group velocity (self-steepening), $\tilde{\Gamma}$ is the intrinsic fibre loss, $\tilde{\delta}$ governs the effects of the third-order linear dispersion, and $\frac{\tau_n}{\tau_0}$, where τ_0 is the normalized input pulsewidth and τ_n is related to the slope of the Raman gain curve (assumed to vary linearly in the vicinity of the mean carrier frequency, ω_0), governs the soliton self-frequency shift (SSFS) effect, [36] and the references therein.

We set the right-hand side of (1.1) equal to zero, we obtain the following equation,

$$iu_\xi + \frac{1}{2}u_{\tau\tau} + |u|^2u + is(|u|^2u)_\tau = 0, \quad (1.2)$$

This equation is related to the Kaup-Newell type of derivative nonlinear Schrödinger equation,

$$iq_t(x,t) = -q_{xx}(x,t) + (\bar{q}q^2)_x \quad (1.3)$$

by change of variables

$$u(\xi,\tau) = q(x,t)e^{i(\frac{t}{4s^4} - \frac{x}{2s^2})}, \quad \xi = \frac{t}{2s^2}, \quad \tau = -\frac{x}{2s} + \frac{t}{2s^3}.$$

And we note that if we replace x by $-x$,equation (1.3) changes into

$$iq_t(x,t) = -q_{xx}(x,t) - (\bar{q}q^2)_x. \quad (1.4)$$

But, we also know if we formulate a Riemann-Hilbert problem for the solution of the inverse spectral problem of the equation (1.4), we find we cannot find solutions of its spectral problem which approach the 2×2 identity matrix $\mathbb{I}$ as $k \to \infty$.It is well-known that there are three kinds of celebrated DNLS equations, including Kaup-Newell equation (i.e Eq.(1.4)), Chen-Lee-Liu equation [37]

$$iq_t + q_{xx} + i|q|^2 q_x = 0,$$

and Gerdjikov-Ivanov(GI) equation [38, 40]

$$iq_t + q_{xx} - iq^2 \bar{q}_x + \frac{1}{2}|q|^4 q = 0 \tag{1.5}$$

It has been found that they may be transformed into each other by gauge transformations [38, 39]. And in [40], the GI-type has the required property of the solutions of its spectral problem which approach the 2×2 identity matrix $\mathbb{I}$ as $k \to \infty$. So,we focus on the GI-type of derivative nonlinear Schrödinger equation. In the following of this paper we also name the GI-type DNLS equation as DNLS equation.

Initial value problems for nonlinear evolution equations with step-like initial data have attracted much attention since the early 1970s [16, 17, 18, 19], but only a few rigorous results concerning the long-time behavior of solutions of such problems were available.In 1980s-1990s, a considerable progress was achieved following the development of the theory of Whitham deformations [20] and the analysis of matrix Riemann-Hilbert problem representations of solutions of initial value problems, see [21, 22, 23] and references therein.Most complete results,obtained by using this approach,were related to integrable equations,for which linear operators from the associated Lax pair were self-adjoint and thus their spectrum was real.In [22],Bikbaev considered the case of the focusing nonlinear Schrödinger equation,which required the development of a much more complicated complex form of the theory of Whitham deformations.

A completely rigorous approach for studying asymptotics of solutions of integrable nonlinear equations was introduced by Deift and Zhou

[9](this approach was inspired by earlier works of Manakov [24] and Its [25];see [10] for a detailed historical review) and further extended by Deift,Venakides,and Zhou [26, 27]. This approach is based on the development of the nonlinear steepest descent method for Riemann-Hilbert problems associated with integrable nonlinear equations. Being originally introduced for studying initial value problems with decaying initial data, this approach was recently adapted by Buckingham and Venakides [28] to problems with shock-type oscillating initial data for focusing nonlinear Schrödinger equation. A central role in this development is played by the so-called $g-$function mechanism allowing to deform the original Riemann-Hilbert problem to a form that can be asymptotically treated with the help of associated singular integral equations.

The Riemann-Hilbert problem approach to initial value problems with nondecaying step-like initial data shares many issues with the adaptation of this approach for studying initial-boundary value problems with non-decaying boundary data [29, 30, 31].However,there is an important difference: in the latter case,the construction of the associated Riemann-Hilbert problem normally requires the knowledge of spectral functions associated with overspecified initial and boundary data,which leads to the fact that results(in particular,the asymptotic results,see [29]) have,in a certain sense,a conditional character.As for the initial value problems of the type considered in this paper,the Riemann-Hilbert construction requires only initial data,and thus,the issue of overdetermination does not arise.

In this paper,we consider a pure step-like initial value problem for the DNLS equation:

$$iq_t + q_{xx} - iq^2\bar{q}_x + \frac{1}{2}|q|^4 q = 0, \qquad x \in \mathbb{R}, t > 0, \tag{1.6a}$$

$$q(x,0) = q_0(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ Ae^{-2iBx} & \text{if } x < 0, \end{cases} \tag{1.6b}$$

where $A > 0$ and $B \in \mathbb{R}$ are some constants. Kitaev and Vartanian got the leading order long-time asymptotic for the KN-type of DNLS equation with the decaying initial value,in [34], and the higher order long-time asymptotic in [36].

Since the DNLS equation (1.6a) has a plane wave solution

$$q^p(x,t) = Ae^{-2iBx+2i\omega t}, \tag{1.7}$$

with

$$\omega := A^2B - 2B^2 + \frac{A^4}{4}, \tag{1.8}$$

which is consistent with (1.6b) for $x < 0$,that is,$q^p(x,0) = q_0(x)$,we assume that the solution $q(x,t)$ of the initial value problem (1.6a) evaluated at any $t > 0$ has the following behavior as $x \to \pm\infty$:

$$q(x,t) = o(1), \qquad x \to +\infty, \tag{1.9}$$

$$q(x,t) = q^p(x,t) + o(1), \qquad x \to -\infty, \tag{1.10}$$

where $o(1)$ means sufficiently fast decay to 0.This assumption can be justified a posteriori,by evaluating the large-x behavior of the solution of the Riemann-Hilbert problem formulated in Section 3.

Recently, in [32],A.Boutet de Monvel,V.P.Kotlyarov, and D.Shepelsky considered the long-time dynamics of the initial value problem for the focusing nonlinear Schrödinger equation with step-like data.The strategy of the Riemann-Hilbert problem deformations that we adopt in this paper is similar,though not identical,to that in [28].In particular,the realization of the $g-$function mechanism is different as well as the resulting asymptotic picture.

As we have already mentioned, the main tool available now for studying rigorously the long-time asymptoitcs of solutions of initial and initial boundary value problems for integrable nonlinear equations is the asymptotic analysis of associated Riemann-Hilbert problems,whose construction involves dedicated solutions of the system of two linear equations,the Lax pair associated with the integrable nonlinear equation.

For the DNLS equation (1.6a), a Lax pair is as follows [34]:

$$\begin{aligned} &\Psi_x(x,t;k) = M(x,t;k)\Psi(x,t;k),\\ &\Psi_t(x,t;k) = N(x,t;k)\Psi(x,t;k), \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} &M(x,t;k) = -ik^2\sigma_3 + kQ + \frac{i}{2}|q|^2\sigma_3,\\ &N(x,t;k) = -2ik^4\sigma_3 + 2k^3Q + ik^2|q|^2\sigma_3 - ikQ_x\sigma_3 + \frac{i}{4}|q|^4\sigma_3 + \frac{1}{2}(q\bar{q}_x - \bar{q}q_x)\sigma_3, \end{aligned} \tag{1.12}$$

with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\Psi(x,t;k)$ is a 2×2 matrix-value function,$k \in \mathbb{C}$ is a spectral parameter, and the matrix coefficient Q is expressed in terms of a scalar function q:

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \tag{1.13}$$

It is well-known [34] that this over-determined system of equations (1.11) is compatible if and only if $q(x,t)$ solves the DNLS equation (1.6a).

In Section 2 we present these dedicated solutions(eigenfunctions) and associated spectral functions.All these functions are then used in Section 3 for constructing a basic Riemann-Hilbert problem,whose solution gives the solution of the initial value problem (1.6a),(1.6b).Section 4 develops the asymptotic analysis of this Riemann-Hilbert problem leading to asymptotic formulas for the solution of the original Cauchy problem (1.6).

2. Eigenfunctions

Let Q^p be defined by (1.13) with q^p instead of q. A particular solution of the system (1.11),with Q^p instead of Q,is given by

$$\Psi^p(x,t;k) = e^{i(\omega t - Bx)\sigma_3}E(k)e^{-i(xX(k)+t\Omega(k))\sigma_3}, \tag{2.1}$$

where

$$X(k)=\sqrt{(k^2-B-\frac{A^2}{2})^2+k^2A^2}, \tag{2.2}$$

$$\Omega(k)=2(k^2+B)X(k). \tag{2.3}$$

$$E(k)=\frac{1}{2}\begin{pmatrix} \varphi(k)+\frac{1}{\varphi(k)} & \varphi(k)-\frac{1}{\varphi(k)} \\ \varphi(k)-\frac{1}{\varphi(k)} & \varphi(k)+\frac{1}{\varphi(k)} \end{pmatrix} \tag{2.4}$$

with

$$\varphi(k)=(\frac{k^2-B-\frac{A^2}{2}-ikA}{k^2-B-\frac{A^2}{2}+ikA})^{\frac{1}{4}}, \tag{2.5}$$

The branch cut for X and φ is taken along the segment

$$\gamma\cup\bar{\gamma}:=\{k\in\mathbb{C}|k_1^2-k_2^2=B,k_1^2\leq C^2\}, \tag{2.6}$$

where $\gamma=\{k\in\mathbb{C}|k_1^2-k_2^2=B,k_1^2\leq C^2,\mathrm{Im}k^2>0\}$, $C^2=B+\frac{A^2}{4}$, $k_1=\mathrm{Re}k$ and $k_2=\mathrm{Im}k$. And the branches are fixed by the asymptotics:

$$X(k)=k^2-B+O(\frac{1}{k^2}),\qquad \text{as } k\to\infty,$$

$$\varphi(k)=1+O(\frac{1}{k}),\qquad \text{as } k\to\infty.$$

We find that $\Omega(k)=2k^4+\omega+O(\frac{1}{k}),$ as $k\to\infty$. We also find that $\mathrm{Im}X(k)=0$ is

$$k_1k_2(k_1^2-k_2^2-B)=0, \tag{2.7}$$

which is on

$$\Sigma:=\mathbb{R}\cup i\mathbb{R}\cup\gamma\cup\bar{\gamma}. \tag{2.8}$$

Thus, for any $t\geq 0$, $\Psi^p(x,t;k)$ is bounded in x if and only if $k\in\Sigma$.

Let $q(x,t)$ be a solution of the Cauchy problem (1.6a),(1.6b) satisfying the asymptotic conditions (1.9),(1.10), and let $Q(x,t)$ and $Q^p(x,t)$ be defined by (1.13), in terms of q and q^p, respectively.Define the 2×2 matrix-value functions $\mu_j(x,t;k)$, $j=1,2$, $-\infty<x<\infty, 0\leq t<\infty$, as the solutions of the Volterra integral equations:

$$\mu_1(x,t;k)=\mathbb{I}+\int_{+\infty}^{x}e^{ik^2(y-x)\sigma_3}(kQ\mu_1)(y,t;k)e^{-ik^2(y-x)\sigma_3},\qquad k^2\in\mathbb{R}, \tag{2.9}$$

$$\begin{aligned}\mu_2(x,t;k) &= e^{i(\omega t-Bx)\sigma_3}E(k) \\ &+\int_{-\infty}^{x}\Gamma^p(x,y,t,k)k[Q-Q^p](y,t)\mu_2(y,t,k)e^{-ik^2(y-x)\sigma_3}, k\in\Sigma,\end{aligned} \tag{2.10}$$

where

$$\Gamma^p(x,y,t,k):=\Psi^p(x,t,k)[\Psi^p(y,t,k)]^{-1}.$$

Note that Γ^p can be written in the form

$$\Gamma^p(x,y,t,k)=e^{i(\omega t-Bx)\sigma_3}G^p(x,y,k)e^{-i(\omega t-By)\sigma_3},$$

where

$$G^p(x,y,k)=\begin{pmatrix}\alpha+i(k^2-B-\frac{A^2}{2})\beta & -kA\beta \\ kA\beta & \alpha-i(k^2-B-\frac{A^2}{2})\beta\end{pmatrix},$$

with

$$\alpha=\cos[(y-x)X(k)],\qquad \beta=\frac{\sin[(y-x)X(k)]}{X(k)}.$$

For any $(x,y)\in\mathbb{R}^2$,$G^p(x,y,k)$ is an entire function of k with asymptotic behavior

$$G^p(x,y,k)=e^{i(y-x)(k^2-B-\frac{A^2}{2})\sigma_3}[\mathbb{I}+O(\frac{1}{k})],\qquad \text{as } k\to\infty,\quad \text{Im}k^2=0.$$

The analytic properties of the 2×2 matrices $\mu_j(x,t;k)$, $j=1,2$, that follow from (2.9) and (2.10) are collected in the following proposition.We denote by $\mu_j^{(1)}(x,t,k)$ and $\mu_j^{(2)}(x,t,k)$ the columns of $\mu_j(x,t;k)$.

Proposition 2.1. *The matrices $\mu_1(x,t;k)$ and $\mu_2(x,t;k)$ have the following properties:*

(i) *$det\mu_1(x,t,k)=\mu_2(x,t;k)=1$.*

(ii) *The functions $\Phi(x,t,k)$ and $\Psi(x,t,k)$ defined by*

$$\Psi(x,t,k):=\mu_1(x,t,k)e^{-ik^2x\sigma_3-2ik^4t\sigma_3},$$

$$\Phi(x,t,k):=\mu_2(x,t;k)e^{-ixX(k)\sigma_3-it\Omega(k)\sigma_3}.$$

satisfy the Lax pair equations (1.11).

(iii) *$\mu_1^{(1)}(x,t,k)$ is analytic in $\text{Im}k^2<0$ and*

$$\mu_1^{(1)}(x,t,k)=\begin{pmatrix}1\\0\end{pmatrix}+O(\frac{1}{k}),\ as\ k\to\infty,\quad \text{Im}k^2\le 0.$$

(iv) $\mu_1^{(2)}(x,t,k)$ *is analytic in* $\mathrm{Im}k^2 > 0$ *and*

$$\mu_1^{(2)}(x,t,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\frac{1}{k}), \text{ as } k \to \infty, \quad \mathrm{Im}k^2 \geq 0.$$

(v) $\mu_2^{(1)}(x,t,k)$ *is analytic in* $\mathrm{Im}k^2 > 0 \backslash \gamma$*,has a jump across* γ*, and*

$$\mu_2^{(1)}(x,t,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(\frac{1}{k}), \text{ as } k \to \infty, \quad \mathrm{Im}k^2 \geq 0.$$

(vi) $\mu_2^{(2)}(x,t,k)$ *is analytic in* $\mathrm{Im}k^2 < 0 \backslash \bar{\gamma}$*,has a jump across* $\bar{\gamma}$*, and*

$$\mu_2^{(2)}(x,t,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\frac{1}{k}), \text{ as } k \to \infty, \quad \mathrm{Im}k^2 \leq 0.$$

(vii) *Moreover,*

$$\mu_j^{(1)}(x,t,k) = \mathbb{I} + \frac{\tilde{\mu}(x,t)}{ik} + o(\frac{1}{k})$$

as $k \to \infty$ *along curves transversal to the real and image axis, where*

$$[\sigma_3, \tilde{\mu}(x,t)] = \begin{pmatrix} 0 & q(x,t) \\ -\bar{q}(x,t) & 0 \end{pmatrix}$$

(viii) $\mu_2^{(2)}(x,t,k)(k-E)^{\frac{1}{4}}$ *is boundary near* $k = E$ *and* $\mu_2^{(2)}(x,t,k)(k-\bar{E})^{\frac{1}{4}}$ *is boundary near* $k = \bar{E}$*.*

Since the eigenfunctions $\Psi(x,t,k)$ and $\Phi(x,t,k)$ satisfy both equations of the Lax pair, we have

$$\Phi(x,t,k) = \Psi(x,t,k)S(k), \qquad k^2 \in \mathbb{R}, \tag{2.11}$$

where $S(k)$ is independent of (x,t). Since (see (2.9) and (2.10) for $t = 0$)

$$\Psi(x,0,k) = e^{-ik^2 x\sigma_3}, \qquad \text{for } x \geq 0,$$

$$\Phi(x,0,k) = e^{-iBx\sigma_3}E(k)e^{-ixX(k)\sigma_3}, \qquad \text{for } x \leq 0,$$

we conclude that

$$S(k) = \Psi^{-1}(0,0,k)\Phi(0,0,k) = \Phi(0,0,k) = E(k). \tag{2.12}$$

Thus, we have

$$S(k)=\begin{pmatrix} \bar{a}(\bar{k}) & b(k) \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix}=\begin{pmatrix} a(k) & b(k) \\ b(k) & a(k) \end{pmatrix}, \tag{2.13}$$

where

$$\begin{aligned} a(k)=\bar{a}(\bar{k})=\frac{1}{2}[\varphi(k)+\frac{1}{\varphi(k)}], \\ b(k)=-\bar{b}(\bar{k})=\frac{1}{2}[\varphi(k)-\frac{1}{\varphi(k)}]. \end{aligned} \tag{2.14}$$

3. The basic Riemann-Hilbert problem

The scattering relation (2.11) involving the eigenfunctions $\Psi(x,t,k)$ and $\Phi(x,t,k)$ can be rewritten in the form of conjugation of boundary values of a piecewise analytic matrix-value function on a contour in the complex $k-$plane,namely:

$$M_+(x,t,k)=M_-(x,t,k)J(x,t,k), \qquad k\in\Sigma, \tag{3.1}$$

where $M_\pm(x,t,k)$ denote the boundary vales of $M(x,t,k)$ according to a chosen orientation of Σ, and $\Sigma=\mathbb{R}\cup i\mathbb{R}\cup\gamma\cup\bar{\gamma}$.

Indeed,let us write (2.11) in the vector form:

$$\begin{aligned} \frac{\Phi^{(1)}(x,t,k)}{a(k)}=\Psi^{(1)}(x,t,k)+r(k)\Psi^{(2)}(x,t,k), \\ \frac{\Phi^{(2)}(x,t,k)}{a(k)}=r(k)\Psi^{(1)}(x,t,k)+\Psi^{(2)}(x,t,k), \end{aligned} \tag{3.2}$$

where

$$r(k):=\frac{b(k)}{a(k)}=\frac{i}{kA}[k^2-B-\frac{A^2}{2}-X(k)], \tag{3.3}$$

and define the matrix $M(x,t,k)$ as follows:

$$M(x,t,k)=\begin{cases} \left(\frac{\Phi^{(1)}(x,t,k)}{a(k)}e^{it\theta(k)} \quad \Psi^{(2)}(x,t,k)e^{-it\theta(k)} \right), & k\in\{k\in\mathbb{C}|\mathrm{Im}k^2>0\backslash\gamma\}, \\ \left(\Psi^{(1)}(x,t,k)e^{it\theta(k)} \quad \frac{\Phi^{(2)}(x,t,k)}{a(k)}e^{-it\theta(k)} \right), & k\in\{k\in\mathbb{C}|\mathrm{Im}k^2<0\backslash\bar{\gamma}\}, \end{cases} \tag{3.4}$$

where

$$\theta(k):=2k^4+\frac{x}{t}k^2, \tag{3.5}$$

Then the boundary values $M_+(x,t,k)$ and $M_-(x,t,k)$ relative to Σ are related by (3.1),where

$$J(x,t,k)=\begin{cases}\begin{pmatrix}1-r^2(k) & -r(k)e^{-2it\theta(k)}\\ r(k)e^{2it\theta(k)} & 1\end{pmatrix}, & k^2\in\mathbb{R},\\ \begin{pmatrix}1 & 0\\ f(k)e^{2it\theta(k)} & 1\end{pmatrix}, & k^2\in\gamma,\\ \begin{pmatrix}1 & f(k)e^{-2it\theta(k)}\\ 0 & 1\end{pmatrix}, & k^2\in\bar{\gamma},\end{cases} \tag{3.6}$$

with

$$f(k):=r_+(k)-r_-(k). \tag{3.7}$$

The jump relation (3.1) considered together with the properties of the eigenfunctions listed in Proposition 1 suggests a way of representing the solution to the Cauchy problem (1.6a) and (1.6b) in terms of the solution of the Riemann-Hilbert problem, which is specified by the initial conditions (1.6b) via the associated spectral function $r(k)$.

The solution $q(x,t)$ of the initial value problem (1.6a) and (1.6b) can be expressed in terms of the solution of the basic Riemann-Hilbert problem as follows:

$$q(x,t)=2i\lim_{k\to\infty}(kM(x,t,k))_{12}. \tag{3.8}$$

where M is the solution of the following Riemann-Hilbert problem:

Basic Riemann-Hilbert problem I.

Given $r(k), k^2\in\mathbb{R}$ and $f(k)=r_+(k)-r_-(k), k^2\in\gamma\cup\bar{\gamma}$, and $\Sigma=\mathbb{R}\cup i\mathbb{R}\cup\gamma\cup\bar{\gamma}$, find a 2×2 matrix-value function $M(x,t,k)$ such that

(i) $M(x,t,k)$ is analytic in $k\in\mathbb{C}\backslash\Sigma$.

(ii) $M(x,t,k)$ is bounded at the end points E and $\bar{E}$.

(iii) The boundary value $M_\pm(x,t,k)$ at Σ satisfy the jump condition

$$M_+(x,t,k)=M_-(x,t,k)J(x,t,k),\quad k\in\Sigma$$

where the jump matrix $J(x,t,k)$ is defined in terms of $r(k)$ and $f(k)$ by (3.6).

(iv) Behavior at ∞

$$M(x,t,k)=\mathbb{I}+O(\frac{1}{k}), \qquad \text{as } k\to\infty.$$

If we try to analysis the long-time asymptotic behavior of the GI-type of DNLS equation (1.6a) and (1.6b) with step-like initial value problem, this type of Riemann-Hilbert problem has a contradiction in the plane wave region. So we try to derive a new Riemann-Hilbert problem, which is similar to the type of nonlinear Schrödinger equation, to overcome this contradiction. That means we arrive at the following Riemann-Hilbert problem.

We define

$$N(x,t,k)=k^{-\frac{\hat{\sigma}_3}{2}}M(x,t,k), \tag{3.9}$$

then the jump condition for N is

$$N_+(x,t,k)=N_-(x,t,k)e^{-i(k^2x+2k^4t)\hat{\sigma}_3}J_N(x,t,k). \tag{3.10}$$

introducing $\lambda=k^2$ and control the branch of k as $Sign\mathrm{Im}k=Sign\mathrm{Im}\lambda$, and define the modified scattering data $\rho(\lambda)=\frac{r(k)}{k}$, [13].

Then

$$X(\lambda)=\sqrt{(\lambda-B-\frac{A^2}{2})^2+\lambda A^2}=\sqrt{(\lambda-B)^2+\frac{A^4}{4}+A^2B}, \tag{3.11}$$

$$\Omega(\lambda)=2(\lambda+B)X(\lambda). \tag{3.12}$$

and the segment

$$\gamma\cup\bar{\gamma}:=\{\lambda\in\mathbb{C}|\lambda_1=B,\lambda_2^2\leq D^2\}, \tag{3.13}$$

where $\gamma=\{k\in\mathbb{C}|\lambda_1=B,\lambda_2^2\leq D^2,\mathrm{Im}\lambda_2>0\}$, $D^2=A^2B+\frac{A^4}{4}$, $\lambda_1=\mathrm{Re}\lambda$ and $\lambda_2=\mathrm{Im}\lambda$. Let $E=B+iD$, then $\gamma=[E,B]$ and $\bar{\gamma}=[B,\bar{E}]$. And the jump condition for N is

$$N_+(x,t,\lambda)=N_-(x,t,\lambda)e^{-i(\lambda x+2\lambda^2t)\hat{\sigma}_3}J_N(x,t,\lambda). \tag{3.14}$$

where

$$J_N(x,t,\lambda)=\begin{cases}\begin{pmatrix}1-\lambda\rho(\lambda)^2 & -\rho(\lambda)e^{-2it\theta(\lambda)}\\ \lambda\rho(\lambda)e^{2it\theta(\lambda)} & 1\end{pmatrix}, & \lambda\in\mathbb{R},\\ \begin{pmatrix}1 & 0\\ \lambda f(\lambda)e^{2it\theta(\lambda)} & 1\end{pmatrix}, & \lambda\in\gamma,\\ \begin{pmatrix}1 & f(\lambda)e^{-2it\theta(\lambda)}\\ 0 & 1\end{pmatrix}, & \lambda\in\bar{\gamma},\end{cases} \tag{3.15}$$

where

$$f(\lambda)=\rho(\lambda)_+-\rho(\lambda)_-. \tag{3.16}$$

In other word,we have the following basic Riemann-Hilbert problem

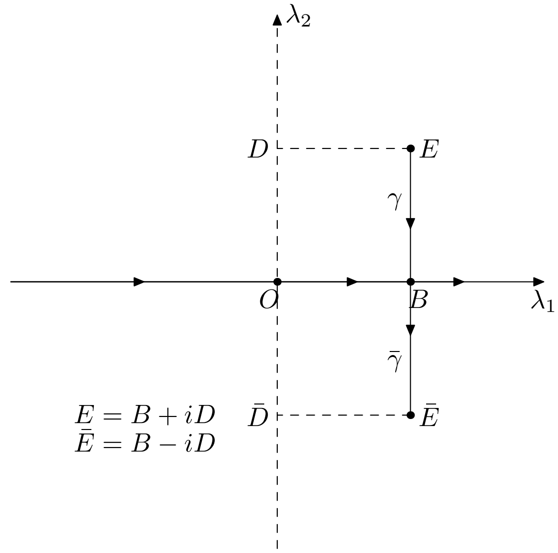

FIGURE 1. The oriented contour $\Sigma=\mathbb{R}\cup\gamma\cup\bar{\gamma}$.

Basic Riemann-Hilbert problem II.

Given $\rho(\lambda),\lambda\in\mathbb{R}$ and $f(\lambda)=\rho(\lambda)_+-\rho(\lambda)_-,\lambda\in\gamma\cup\bar{\gamma}$, and $\Sigma=\mathbb{R}\cup\gamma\cup\bar{\gamma}$, find a 2×2 matrix-value function $N(x,t,\lambda)$ such that

(i) $N(x,t,\lambda)$ is analytic in $\lambda\in\mathbb{C}\backslash\Sigma$.

(ii) $N(x,t,\lambda)$ is bounded at the end points E and $\bar{E}$.

(iii) The boundary value $N_\pm(x,t,\lambda)$ at Σ satisfy the jump condition

$$N_+(x,t,\lambda)=N_-(x,t,\lambda)J_N(x,t,\lambda),\quad \lambda\in\Sigma\backslash\{E,\bar{E},B\},$$

where the jump matrix $J_N(x,t,k)$ is defined in terms of $\rho(\lambda)$ and $f(\lambda)$ by (3.15).

(iv) Behavior at ∞

$$N(x,t,\lambda)=\mathbb{I}+O(\frac{1}{\lambda}),\qquad \text{as } \lambda\to\infty.$$

4. Long-time Asymptotics

The representation of the solution $q(x,t)$ of the initial value problem (1.6) in terms of the solution of an associated basic Riemann-Hilbert problem allows using the ideas of the asymptotic analysis of oscillating Riemann-Hilbert problems [9, 28, 10, 11, 32] for studying the long-time asymptotics of $q(x,t)$. The key fact leading to different asymptotics in different regions of the (x,t) half-plane is that the behavior of the jump matrix of the basic Riemann-Hilbert problem as a function of the large parameter t is different in these regions. Indeed, as seen on (3.15), this behavior is governed by the sign of $\mathrm{Im}\theta(\lambda)$, which itself depends on $\xi=\frac{x}{4t}$. As we have already written, three regions are to be distinguished:

(i) A Zakharov-Manakov region:$\xi>-B$.
(ii) A plane wave region:$\xi<-\sqrt{2}D-B$.
(iii) An elliptic wave region:$-\sqrt{2}D-B<\xi<-B$.

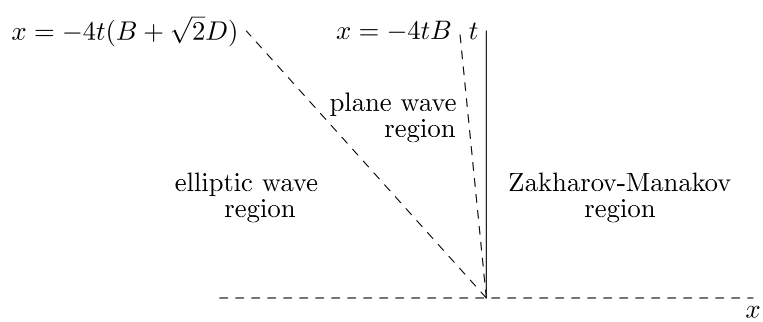

FIGURE 2. The different regions of the $(x,t)-$plane.

4.1. **The Zakharov-Manakov region:**$\xi > -B$**.** In this region $\xi > -B$, we have $\mathrm{Im}\theta(\lambda) > 0$ for all $\lambda \in \gamma$ and $\mathrm{Im}\theta(\lambda) < 0$ for all $\lambda \in \bar{\gamma}$. Therefore, the exponentials in the jump matrix J_N, see (3.15), are decaying as $t \to +\infty$ for $\lambda \in \Sigma \backslash \mathbb{R}$.

This implies that one can follow the technique of asymptotic analysis proposed for the first time in [9]. The basic step of the procedure is a deformation of the original Riemann-Hilbert problem, with the help of the solution of an appropriate scalar Riemann-Hilbert problem, in order to obtain an equivalent Riemann-Hilbert problem whose jump matrix decays, in t, to a constant (in λ) matrix. This leads to model Riemann-Hilbert problems whose solutions can be given explicitly.

A particular feature of the Riemann-Hilbert problem under consideration is that the contour of the modified Riemann-Hilbert problem contains neither the real axis, where the jump matrix for the original Riemann-Hilbert problem oscillates with t, see (3.15), nor the finite parts γ and $\bar{\gamma}$. This happens due to the pure step-like initial conditions, which in turn implies that the associated spectral functions $\rho(\lambda)$ and $\lambda\rho(\lambda)$ can be analytically extended from the contour to the whole λ-plane.

4.1.1. *First transformation.* The first transform is as usual:

$$N^{(1)}(x,t,\lambda) = N(x,t,\lambda)\delta^{-\sigma_3}(\lambda), \tag{4.1}$$

where ([41])

$$\delta(\lambda) = \exp \frac{1}{2\pi i} \int_{-\infty}^{\lambda_0} \frac{\log\left(1-\lambda'\rho(\lambda')^2\right)}{\lambda'-\lambda} d\lambda', \tag{4.2}$$

is the solution of the following scalar Riemann-Hilbert problem:

- $\delta(\lambda)$ is analytic in $\mathbb{C}\backslash(-\infty, \lambda_0]$,
- $\delta(\lambda) \to 1$ as $\lambda \to \infty$,
- $\delta(\lambda)$ satisfies the jump relation

$$\delta_+(\lambda) = \delta_-(\lambda)(1-\lambda\rho^2(\lambda)), \qquad \lambda \in (-\infty, \lambda_0). \tag{4.3}$$

Here, λ_0 is the stationary point of the phase function $\theta(\lambda) = 2\lambda^2 + 4\xi\lambda$, that is, $\theta'(\lambda_0) = 0$:

$$\lambda_0 = -\xi = \frac{-x}{4t}.$$

Then $N^{(1)}(x,t,\lambda)$ satisfies the jump condition

$$\begin{aligned} &N_+^{(1)}(x,t,\lambda) = N_-^{(1)}(x,t,N)J_N^{(1)}(x,t,\lambda), \\ &\lambda \in \Sigma^{(1)} = \Sigma, \end{aligned} \tag{4.4}$$

where

$$J_N^{(1)}(x,t,\lambda) = \delta_-^{\sigma_3} J_N \delta_+^{-\sigma_3},$$

that is

$$J_N^{(1)}(x,t,\lambda) = \begin{cases} e^{-it\theta\hat{\sigma}_3}\begin{pmatrix} \frac{\delta_-}{\delta_+}(1-\lambda\rho(\lambda)^2) & -\rho\delta_+\delta_- \\ \frac{\lambda\rho}{\delta_+\delta_-} & \frac{\delta_+}{\delta_-} \end{pmatrix}, & \lambda \in \mathbb{R}, \\ \begin{pmatrix} \frac{\delta_-}{\delta_+} & 0 \\ \frac{\lambda f}{\delta_+\delta_-}e^{2it\theta\sigma_3} & \frac{\delta_+}{\delta_-} \end{pmatrix}, & \lambda \in \gamma, \\ \begin{pmatrix} \frac{\delta_-}{\delta_+} & f\delta_+\delta_- e^{-2it\theta\sigma_3} \\ 0 & \frac{\delta_-}{\delta_+} \end{pmatrix}, & \lambda \in \bar{\gamma}. \end{cases} \tag{4.5}$$

From the Riemann-Hilbert problem of the δ, we can find

$$J_N^{(1)}(x,t,\lambda) = \begin{cases} e^{-it\theta\hat{\sigma}_3}\begin{pmatrix} 1-\lambda\rho^2 & -\rho\delta^2 \\ \frac{\lambda\rho}{\delta^2} & 1 \end{pmatrix}, & \lambda > \lambda_0, \\ e^{-it\theta\hat{\sigma}_3}\begin{pmatrix} 1 & \frac{-\rho}{1-\lambda\rho^2}\delta_-^2 \\ \frac{\lambda\rho}{1-\lambda\rho^2}\frac{1}{\delta_+^2} & 1-\lambda\rho^2 \end{pmatrix}, & \lambda < \lambda_0, \\ \begin{pmatrix} 1 & 0 \\ \frac{\lambda f}{\delta^2}e^{2it\theta} & 1 \end{pmatrix}, & \lambda \in \gamma, \\ \begin{pmatrix} 1 & f\delta^2 e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & \lambda \in \bar{\gamma}. \end{cases} \tag{4.6}$$

4.1.2. *Second transformation.* The next transformation is:

$$N^{(2)}(x,t,\lambda) = N^{(1)}(x,t,\lambda)G(\lambda), \tag{4.7}$$

where

$$G(\lambda)=\begin{cases}\begin{pmatrix}1 & \frac{\rho}{1-\lambda\rho^2}\delta_-^2e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & \lambda\in D_1,\\ \begin{pmatrix}1 & 0\\ \frac{\lambda\rho}{1-\lambda\rho^2}\frac{1}{\delta_+^2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in D_2,\\ \begin{pmatrix}1 & -\rho\delta^2e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & \lambda\in D_3,\\ \begin{pmatrix}1 & 0\\ \frac{-\lambda\rho}{\delta^2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in D_4,\\ \mathbb{I}, & \lambda\in D_5\cup D_6.\end{cases} \tag{4.8}$$

The domains $D_1,\ldots D_6$ are shown on the following Figure.

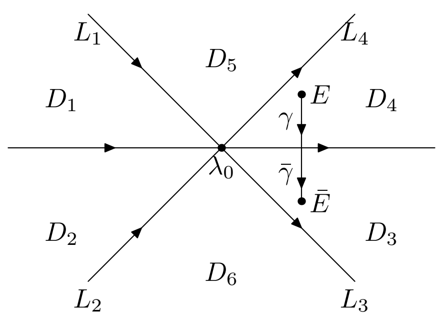

FIGURE 3. The oriented contour $\Sigma^{(2)}=L_1\cup L_2\cup L_3\cup L_4$.

This new function $N^{(2)}$ solves the equivalent Riemann-Hilbert problem:

$$N_+^{(2)}(x,t,\lambda)=N_-^{(2)}(x,t,\lambda)J_N^{(2)}(x,t,\lambda),$$
$$\lambda\in\Sigma^{(2)},$$

where

$$J_N^{(2)}(x,t,\lambda)=\begin{cases}\begin{pmatrix}1 & \frac{-\rho}{1-\lambda\rho^2}\delta_-^2e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & \lambda\in L_1,\\ \begin{pmatrix}1 & 0\\ \frac{\lambda\rho}{1-\lambda\rho^2}\frac{1}{\delta_+^2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in L_2,\\ \begin{pmatrix}1 & -\rho\delta^2e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & \lambda\in L_3,\\ \begin{pmatrix}1 & 0\\ \frac{\lambda\rho}{\delta^2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in L_4.\end{cases} \tag{4.9}$$

4.1.3. *The last transformation.* Now $J_N^{(2)}(x,t,\lambda)$ decays exponentially fast to the identity matrix, as $t\to+\infty$, and uniformly outside any neighborhood of $\lambda=\lambda_0$. Thus, we are in a situation where the asymptotic analysis of [41] works. Particularly,

$$N^{(2)}(x,t,\lambda)=Z(x,t,\lambda)N^{as}(x,t,\lambda),$$

where $N^{as}(x,t,\lambda)$ is a solution of the model problem explicitly given in terms of parabolic cylinder functions whereas $Z(x,t,\lambda)$ can be estimated:

$$Z(x,t,\lambda)=\mathbb{I}+O(\frac{logt}{t^{\frac{1}{2}}}).$$

Therefore, the final asymptotic result is as in [41] giving the main term of the asymptotic in terms of the modified reflection coefficient $\rho(\lambda)$:

Theorem 4.1. *(The Zakharov-Manakov region) In the region $x>-4tB$, the asymptotics, as $t\to+\infty$, of the solution $q(x,t)$ of the initial value problem (1.6) is described by the Zakharov-Manakov type formula*

$$q(x,t)=q_{as}(x,t)+O(\frac{\log t}{t}) \tag{4.10}$$

where

$$
\begin{aligned}
&q_{as} = \tfrac{1}{\sqrt{t}}\alpha(\lambda_0)e^{\frac{ix^2}{4t}-i\nu(\lambda_0)\log t},\\
&|\alpha(\lambda_0)|^2 = \tfrac{\nu(\lambda_0)}{2} = -\tfrac{1}{4\pi}\log(1-\lambda_0|\rho(\lambda_0)|^2),\\
&\arg\alpha(\lambda_0) = -3\nu\log 2 - \tfrac{\pi}{4} + \arg\Gamma(i\nu) - \arg r(\lambda_0) + \tfrac{1}{\pi}\int_{-\infty}^{\lambda_0}\log|\lambda-\lambda_0|d\log(1-\lambda|\rho(\lambda)|^2),\\
&\lambda_0 = -\tfrac{x}{4t}.
\end{aligned}
\tag{4.11}
$$

4.2. **The plane wave region:** $\xi < -\sqrt{2}D - B$. For $x < -4t(B + \sqrt{2A^2(B+\frac{A^2}{4})})$, that means, $\mathrm{Im}\theta(\lambda)$ is negative on γ and positive on $\bar{\gamma}$, which implies that the exponentials in (3.15) increase with t. Thus, the jump matrix J_N for the Riemann-Hilbert problem does not converge to a reasonable limit as $t \to \infty$.

To bypass this difficulty, one deforms the Riemann-Hilbert problem in such a way that the phase $\mathrm{Im}\theta(\lambda)$ is replaced by another function, $g(\lambda)$, providing suitable behavior of the modified jump matrix. The extension of the nonlinear steepest descent method for Riemann-Hilbert problems, involving the g-function mechanism was first proposed by Deift, Venakides, and Zhou, see [26, 27].

4.2.1. *The g function.* A natural choice for a g-function appropriate for the region adjacent to the half-axis $x < 0$, $t = 0$, is the phase appearing in the explicit expression for the eigenfunction Ψ^p, see (2.1), associated with the potential q^p. Setting

$$g(x,t,\lambda) = xX(\lambda) + t\Omega(\lambda), \tag{4.12}$$

where $X(\lambda)$ and $\Omega(\lambda)$ are defined in (3.11) and (3.12),we have

$$\Psi^p(x,t,k) = e^{i(\omega t - Bx)\sigma_3}E(\lambda)e^{-ig(x,t,\lambda)\sigma_3} \tag{4.13}$$

The signature table for $\mathrm{Im}g(\lambda;\xi)$ is the partition of the λ-plane into maximal domains where the sign of $\mathrm{Im}g(\lambda;\xi)$ is constant. Its form can be controlled by the zeros of the differential $dg(\lambda)$. Indeed,

$$dg(\lambda) = 4\frac{(\lambda-\mu_+)(\lambda-\mu_-)}{X(\lambda)}d\lambda, \tag{4.14}$$

where

$$\mu_\pm = \frac{B-\xi}{2} \pm \sqrt{\frac{(B+\xi)^2}{4} - \frac{\frac{A^4}{4}+A^2B}{2}}, \tag{4.15}$$

Thus, for $\xi < -(B + \sqrt{2A^2(B+\frac{A^2}{4})})$, $\mu_\pm$ are both real. Moreover,

$$B < \mu_- < \mu_+ < -\xi.$$

In what follows the signature table of the function Im$g(\lambda)$ for different values of ξ plays a very important role. The lines of separation between the different domains are the real axile

$$\lambda_2 = 0,$$

and the algebraic curve

$$\lambda_2^2(\lambda_1+\xi) = (\lambda_1+B+2\xi)[(\lambda_1-B)(\lambda_1+\xi) + \frac{\frac{A^4}{4}+A^2B}{2}], \tag{4.16}$$

They are indeed given by Im$g(\lambda) = 0$. Because of

$$\mathrm{Im}g(\lambda) = 4\lambda_2\{(\lambda_1+B+2\xi)[(\lambda_1-B)(\lambda_1+\xi)+\frac{\frac{A^4}{4}+A^2B}{2}]-\lambda_2^2(\lambda_1+\xi)\}$$

The equation (4.16) can be written:

$$\lambda_2^2(\lambda_1+\xi) = (\lambda_1+B+2\xi)[(\lambda_1-\mu_+)(\lambda_1-\mu_-)].$$

And the signature table of the function Im$g(\lambda)$ is shown in the following Figure 4.

The advantage of the signature table shown in Figure 4 is that there is a finite arc connecting the branch points E and $\bar{E}$ such that Im$g(\lambda) = 0$ for all λ along this arc. Since the jump matrix depends on t via exponentials of type $e^{\pm ig(\lambda)}$, it is oscillatory along an arc where Im$g(\lambda) = 0$.

This suggests to deform the original contour $\gamma \cup \bar{\gamma}$ of the basic Riemann-Hilbert problem to a new contour $\gamma_g \cup \bar{\gamma}_g$ which depends on ξ and where Im$g(\lambda) = 0$, and to view $X(\lambda)$, thus also $g(\lambda)$ as functions with branch cut $\gamma_g \cup \bar{\gamma}_g$.

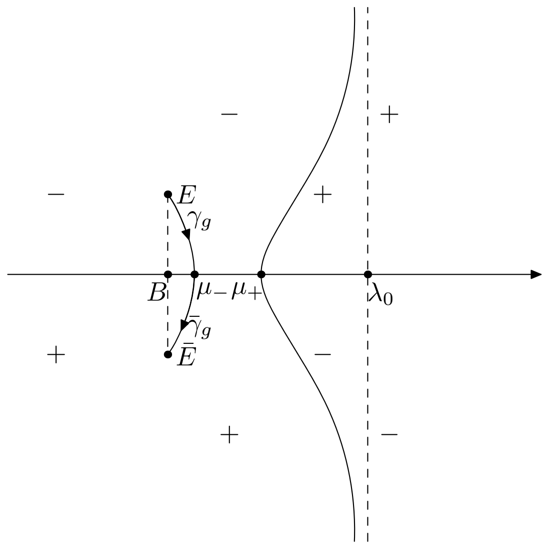

FIGURE 4. The curves of $\mathrm{Im}g(\lambda) = 0$ for $x < -4t(B + \sqrt{2A^2(B + \frac{A^2}{4}}))$.

Another important feature of $g(\lambda;\xi)$ is that it has, up to a constant, the same large λ asymptotic behavior as the phase function $\theta(\lambda)$:

$$g(\lambda;\xi) = t(2\lambda^2 + 4\xi\lambda + g(\infty;\xi)) + O(\frac{1}{\lambda}), \qquad \lambda \to \infty, \tag{4.17}$$

where

$$g(\infty;\xi) = (\omega - 4B\xi). \tag{4.18}$$

4.2.2. *The first transformation.* We put

$$N^{(1)}(x,t,\lambda) = e^{-itg(\infty,\xi)\sigma_3} N(x,t,\lambda) e^{-i(\lambda x + 2\lambda^2 t - g(\lambda))\sigma_3},$$

Then the matrix-value function $N^{(1)}(x,t,\lambda)$ satisfies the following Riemann-Hilbert problem:

$$N_+^{(1)}(x,t,\lambda) = N_-^{(1)}(x,t,\lambda) J_N^{(1)}(x,t,\lambda), \qquad \lambda \in \Sigma^{(1)} = \mathbb{R} \cup \gamma_g \cup \bar{\gamma}_g,$$

with the jump matrix

$$J_N^{(1)}(x,t,\lambda)=\begin{cases}\begin{pmatrix}1-\lambda\rho^2(\lambda) & -\rho(\lambda)e^{-2ig(\lambda)}\\ \lambda\rho(\lambda)e^{2ig(\lambda)} & 1\end{pmatrix}, & \lambda\in\mathbb{R},\\ \begin{pmatrix}e^{-2ig_-(\lambda)} & 0\\ \lambda f(\lambda) & e^{2ig_-(\lambda)}\end{pmatrix}, & \lambda\in\gamma_g,\\ \begin{pmatrix}e^{-2ig_-(\lambda)} & f(\lambda)\\ 0 & e^{2ig_-(\lambda)}\end{pmatrix}, & \lambda\in\bar{\gamma}_g.\end{cases}\tag{4.19}$$

Here $g_\pm(\lambda)$ are boundary values of g on $\gamma_g\cup\bar{\gamma}_g$, and they are real. We also use the equation $g_+(\lambda)=-g_-(\lambda)$.

4.2.3. *The second transformation.* The next transformation is similar to the first transformation applied in the ZakharovManakov region, see Section 4.1.1. It involves the solution $\delta(\lambda)$ of the scalar Riemann-Hilbert problem 4.3) but with μ_+ instead of λ_0,where μ_+ is the stationary point of the new phase function $g(\lambda)$. With this new scalar function $\delta(\lambda)$, we set

$$N^{(2)}(x,t,\lambda)=N^{(1)}(x,t,\lambda)\delta^{-\sigma_3}(\lambda),$$

Then the matrix-value function $N^{(2)}(x,t,\lambda)$ satisfies the following Riemann-Hilbert problem

$$N_+^{(2)}(x,t,\lambda)=N_-^{(2)}(x,t,\lambda)J_N^{(2)}(x,t,\lambda),\qquad \lambda\in\Sigma^{(2)}=\Sigma^{(1)},\tag{4.20}$$

where $J_N^{(2)}(x,t,\lambda)$ is defined as follows:

$$J_N^{(2)}(x,t,\lambda)=\begin{cases}e^{-ig\hat{\sigma}_3}\begin{pmatrix}1-\lambda\rho^2 & -\rho\delta^2\\ \frac{\lambda\rho}{\delta^2} & 1\end{pmatrix}, & \lambda>\mu_+,\\ e^{-ig\hat{\sigma}_3}\begin{pmatrix}1 & \frac{-\rho}{1-\lambda\rho^2}\delta_-^2\\ \frac{\lambda\rho}{1-\lambda\rho^2}\frac{1}{\delta_+^2} & 1-\lambda\rho^2\end{pmatrix}, & \lambda<\mu_+,\\ \begin{pmatrix}e^{-2ig_-(\lambda)} & 0\\ \frac{\lambda f}{\delta^2} & e^{2ig_-(\lambda)}\end{pmatrix}, & \lambda\in\gamma_g,\\ \begin{pmatrix}e^{-2ig_-(\lambda)} & f\delta^2\\ 0 & e^{2ig_-(\lambda)}\end{pmatrix}, & \lambda\in\bar{\gamma}_g.\end{cases}\tag{4.21}$$

4.2.4. *The third transformation.* The subsequent transformation

$$N^{(3)}(x,t,\lambda)=N^{(2)}(x,t,\lambda)G(\lambda),$$

involves $G(\lambda)$ defined similarly to (4.8), with $t\theta$ replaced by g and λ_0 replaced by μ_+. Then $N^{(3)}(x,t,\lambda)$ satisfies the jump relation

$$N_+^{(3)}(x,t,\lambda)=N_-^{(3)}(x,t,\lambda)J_N^{(3)}(x,t,\lambda),$$

across to the contour

$$\Sigma^{(3)}=L_1\cup L_2\cup L_3\cup L_4\cup\gamma_g\cup\bar{\gamma}_g,$$

shown in Figure 5.

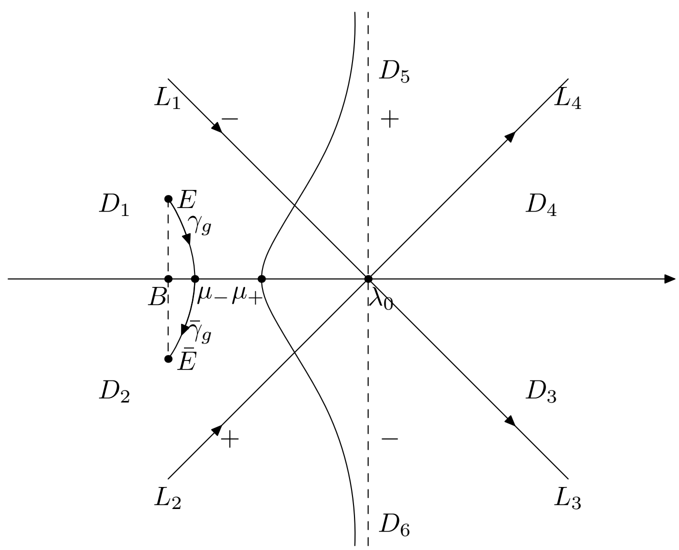

FIGURE 5. The contour $\Sigma^{(3)}=L_1\cup L_2\cup L_3\cup L_4\cup\gamma_g\cup\bar{\gamma}_g$ of the Riemann-Hilbert problem for $N^{(3)}$ for $x<-4t(B+\sqrt{2A^2(B+\frac{A^2}{4})})$.

And we notice that

1.For $\lambda\in L_1\cup L_2\cup L_3\cup L_4$ the jump matrix $J_N^{(3)}(x,t,\lambda)$ decays to the identity matrix, as $t\to\infty$, exponentially fast and uniformly outside any neighborhood of $\lambda=\mu_+$.

2.For $\lambda \in \gamma_g$, the jump matrix $J_N^{(3)}(x,t,\lambda)$ factorizes as

$$\begin{pmatrix} 1 & (\frac{-\rho}{1-\lambda\rho^2})_-\delta^2 e^{-2ig_-(\lambda)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-2ig_(\lambda)} & 0 \\ \lambda f(\lambda)\delta^{-2}(\lambda) & e^{2ig_-(\lambda)} \end{pmatrix} \begin{pmatrix} 1 & (\frac{\rho}{1-\lambda\rho^2})_+\delta^2 e^{2ig_-(\lambda)} \\ 0 & 1 \end{pmatrix} \tag{4.22}$$

3.For $\lambda \in \bar{\gamma}_g$, the jump matrix $J_N^{(3)}(x,t,\lambda)$ factorizes as

$$\begin{pmatrix} 1 & 0 \\ (\frac{-\lambda\rho}{1-\lambda\rho^2})_-\delta^{-2} e^{2ig_-(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} e^{-2ig_-(\lambda)} & f(\lambda)\delta^2(\lambda) \\ 0 & e^{2ig_-(\lambda)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (\frac{\lambda\rho}{1-\lambda\rho^2})_+\delta^{-2} e^{2ig_-(\lambda)} & 1 \end{pmatrix} \tag{4.23}$$

4.Using the identities

$$1 + \lambda f(\frac{-\rho}{1-\lambda\rho^2})_- = 0,$$

$$1 + f(\frac{\lambda\rho}{1-\lambda\rho^2})_+ = 0,$$

we find

$$J_N^{(3)}(x,t,k) = \begin{cases} \begin{pmatrix} 0 & -(\lambda f)^{-1}(\lambda)\delta^2(\lambda) \\ \lambda f(\lambda)\delta^{-2}(\lambda) & 0 \end{pmatrix}, & \lambda \in \gamma_g, \\ \begin{pmatrix} 0 & f(\lambda)\delta^2(\lambda) \\ -f^{-1}(\lambda)\delta^{-2}(\lambda) & 0 \end{pmatrix}, & \lambda \in \bar{\gamma}_g, \end{cases} \tag{4.24}$$

In order to arrive at a Riemann-Hilbert problem whose jump matrix does not depend on λ, we introduce a factorization involving a scalar function $F(\lambda)$ to be defined;

$$J_N^{(3)}(x,t,\lambda) = \begin{pmatrix} F_+^{-1}(\lambda) & 0 \\ 0 & F_+(\lambda) \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} F_-(\lambda) & 0 \\ 0 & F_-^{-1}(\lambda) \end{pmatrix}, \tag{4.25}$$

in such a way that the boundary values $F_\pm(\lambda)$ of $F(\lambda)$ along the two sides of $\gamma_g \cup \bar{\gamma}_g$ satisfy

$$F_-(\lambda)F_+(\lambda) = \begin{cases} -i\lambda f(\lambda)\delta^{-2}(\lambda) & \lambda \in \gamma_g, \\ if^{-1}(\lambda)\delta^{-2}(\lambda) & \lambda \in \bar{\gamma}_g. \end{cases} \tag{4.26}$$

Indeed, once (4.25) is satisfied, one can absorb the diagonal factors into a new piecewise analytic function whose jump across $\gamma_g \cup \bar{\gamma}_g$ is only the constant middle factor in (4.25).

Thus, we arrive at the following scalar Riemann-Hilbert problem:

Scalar Riemann-Hilbert problem.

Find a scalar function $F(\lambda)$ such that

- $F(\lambda)$ and $F^{-1}(\lambda)$ are analytic in $\mathbb{C}\backslash\{\gamma_g \cup \bar{\gamma}_g\}$.
- $F(\lambda)$ satisfies the jump relation:

$$F_+(\lambda)F_-(\lambda) = \begin{cases} -i\lambda f(\lambda)\delta^{-2}(\lambda) = a_+^{-1}(\lambda)a_-^{-1}(\lambda)\sqrt{\lambda}\delta^{-2}(\lambda), & \lambda \in \gamma_g, \\ if^{-1}(\lambda)\delta^{-2}(\lambda) = a_+(\lambda)a_-(\lambda)\sqrt{\lambda}\delta^{-2}(\lambda), & \lambda \in \bar{\gamma}_g. \end{cases} \tag{4.27}$$

  where the contour $\gamma_g \cup \bar{\gamma}_g$ is oriented from E to $\bar{E}$, and
- $F(\lambda)$ is bounded at $\lambda = \infty$.

Introducing

$$H(\lambda) = \begin{cases} F(\lambda)a(\lambda), & \lambda \in \mathbb{C}_+\backslash\gamma_g, \\ \frac{F(\lambda)}{a(\lambda)}, & \lambda \in \mathbb{C}_-\backslash\bar{\gamma}_g. \end{cases} \tag{4.28}$$

then the jump relation (4.27) transforms to

$$[\frac{\log H(\lambda)}{X(\lambda)}]_+ - [\frac{\log H(\lambda)}{X(\lambda)}]_- = \begin{cases} \frac{\log\sqrt{\lambda}\delta^{-2}(\lambda)}{X(\lambda)_+}, & \lambda \in \gamma_g \cup \bar{\gamma}_g, \\ \frac{\log a^2(\lambda)}{X(\lambda)}, & \lambda \in \mathbb{R}. \end{cases} \tag{4.29}$$

The Sokhotski-Plemelj formula shows that this last jump relation is satisfied by

$$H(k) = \exp\{\frac{X(\lambda)}{2\pi i}[\int_{\gamma_g\cup\bar{\gamma}_g} \frac{\log\sqrt{s} + \log\delta^{-2}(s,\xi)}{s-\lambda}\frac{ds}{X_+(s)} + \int_{\mathbb{R}} \frac{\log ab(s)}{s-\lambda}\frac{ds}{X(s)}]\} \tag{4.30}$$

Then $F(\lambda)$ is defined in terms of $H(\lambda)$ by (4.28). At $\lambda = \infty$ we find

$$F(\infty) = H(\infty) = e^{i\phi(\xi)},$$

where

$$\phi(\xi) = \frac{1}{2\pi}[\int_{\gamma_g\cup\bar{\gamma}_g} \frac{\log\sqrt{s}\delta^{-2}(s,\xi)}{X_+(s)}ds + \int_{\mathbb{R}} \frac{\log a^2(s)}{X(s)}ds] \tag{4.31}$$

with

$$\delta(\lambda,\xi)=\exp\frac{1}{2\pi i}\int_{-\infty}^{\mu_+}\frac{\log(1-\lambda'\rho(\lambda')^2)}{\lambda'-\lambda}d\lambda', \tag{4.32}$$

Using the relation $1-\lambda\rho^2(\lambda)=a^{-2}(\lambda)$, we find a simpler expression for $\phi(\xi)$:

$$\phi(\xi)=\frac{1}{2\pi}[\int_{\mu_+}^{+\infty}\log a^2(\lambda)\frac{d\lambda}{X(\lambda)}+\int_{\gamma_g\cup\bar{\gamma}_g}\frac{\log\sqrt{\lambda}}{X_+(\lambda)}d\lambda]$$

4.2.5. *The fourth transformation.* The factorization (4.25) suggests a fourth transformation

$$N^{(4)}(x,t,\lambda)=F^{\sigma_3}(\infty,\xi)N^{(3)}(x,t,\lambda)F^{-\sigma_3}(\lambda,\xi),$$

Then we have

$$N_+^{(4)}(x,t,\lambda)=N_-^{(4)}(x,t,\lambda)J_N^{(4)}(x,t,\lambda)$$

For $\lambda\in\gamma_g\cup\bar{\gamma}_g$ the jump matrix $J_N^{(4)}(x,t,\lambda)$ is constant

$$J_N^{(4)}(x,t,\lambda)=J_N^{mod}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

1.For $\lambda\in\gamma_g\cup\bar{\gamma}_g$ the jump matrix $J_N^{(4)}(x,t,\lambda)$ is constant:

$$J_N^{(4)}(x,t,\lambda)=J_N^{mod}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

2.For $\lambda\in L\cup\bar{L}$, the jump matrix $J_N^{(4)}(x,t,\lambda)$ decays to the identity

$$J_N^{(4)}(x,t,\lambda)=\mathbb{I}+O(\frac{1}{e^{\varepsilon t}}).$$

4.2.6. *The final transformation.* Finally, we can express $N^{(4)}$ in the form

$$N^{(4)}(x,t,\lambda)=N^{err}(x,t,\lambda)N^{mod}(x,t,\lambda),$$

where $N^{mod}(x,t,\lambda)$ solves the model problem:

$$N_-^{mod}(x,t,\lambda)=N_+^{(mod)}(x,t,\lambda)J_N^{mod},\qquad \lambda\in\gamma_g\cup\bar{\gamma}_g, \tag{4.33}$$

with constant jump matrix

$$J_N^{mod}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},$$

and $N^{err}(x,t,\lambda) = \mathbb{I} + O(t^{-\frac{1}{2}})$.

As for the model problem, since $\varphi(\lambda)_- = i\varphi(\lambda)_+$ on $\gamma_g \cup \bar{\gamma}_g$, its solution can be given explicitly in terms of $\varphi(\lambda)$:

$$N^{mod}(x,t,\lambda) = \frac{1}{2}\begin{pmatrix} \varphi(\lambda) + \frac{1}{\varphi(\lambda)} & \varphi(\lambda) - \frac{1}{\varphi(\lambda)} \\ \varphi(\lambda) - \frac{1}{\varphi(\lambda)} & \varphi(\lambda) + \frac{1}{\varphi(\lambda)} \end{pmatrix}.$$

4.2.7. *Back to the original problem.* Let $N^*(x,t,\lambda)$, $* =$ (1),(2),(3),(4),mod, denote the solution of the Riemann-Hilbert problem RH^*, and let

$$m_{12}^*(x,t) = \lim_{\lambda\to\infty} (\lambda M^*(x,t,\lambda))_{12},$$

Then, going back to the determination of $q(x,t)$ in terms of the solution of the basic Riemann-Hilbert problem, we have

$$\begin{aligned} q(x,t) =& 2im(x,t)_{12} = 2ie^{2ig(\infty,\xi)}m^{(1)}(x,t)_{12} \\ =& 2ie^{2ig(\infty,\xi)}m^{(2)}(x,t)_{12} + O(t^{-\frac{1}{2}}) \\ =& 2ie^{2ig(\infty,\xi)}m^{(3)}(x,t)_{12} + O(t^{-\frac{1}{2}}) \\ =& 2ie^{2ig(\infty,\xi)}m^{(4)}(x,t)_{12}F^{-2}(\infty,\xi) + O(t^{-\frac{1}{2}}) \\ =& 2ie^{2ig(\infty,\xi)}m^{mod}(x,t)_{12}F^{-2}(\infty,\xi) + O(t^{-\frac{1}{2}}). \end{aligned} \tag{4.34}$$

Taking into account that $g(\infty,\xi) = \omega t - 4Bx$, $2im^{mod}(x,t)_{12} = A$ and $F^{-2}(\infty,\xi) = e^{-2i\phi(\xi)}$we arrive at the following theorem:

Theorem 4.2. *(***Plane wave region***) In the region $x < -4t(B + \sqrt{2A^2(B+\frac{A^2}{4})})$,the asymptotics, as $t\to+\infty$, of the solution $q(x,t)$ of the initial value problem (1.6) takes the form of a plane wave:*

$$q(x,t) = Ae^{2i(\omega t - Bx - \phi(\xi))} + O(t^{-\frac{1}{2}}), \qquad t\to+\infty. \tag{4.35}$$

Remark 4.3. *If we let $\xi\to+\infty$, then $\mu_+\to+\infty$, then $\phi(\xi)\to\phi$, with $\phi = \frac{1}{2\pi}\int_{\gamma_g\cup\bar{\gamma}_g}\frac{\log\sqrt{\lambda}}{X_+(\lambda)}d\lambda$, and then the above equation (4.61) reduce to $q(x,t) = Ae^{2i(\omega t - Bx - \phi)}$, this is correspondence to our initial condition up to a phase shift.*

4.3. **The elliptic region:**$-4t(B+\sqrt{2}D) < x < -4tB$**.** For the limit case $\xi_0 = -(B+\sqrt{2A^2(B+\frac{A^2}{4})})$, we have $\mu_+(\xi_0) = \mu_-(\xi_0)$, see Figure 7, whereas for $\xi > -(B+\sqrt{2A^2(B+\frac{A^2}{4})})$, μ_+ and μ_- become non-real, complex conjugated numbers. As a result, the g-function mechanism with $g(\lambda;\xi)$ as in the plane wave region fails. This shows that there is a break in the qualitative picture of the asymptotic behavior at $\xi = \xi_0$.

4.3.1. *The new g-function.* A suitable g-function for $\xi > -(B+\sqrt{2A^2(B+\frac{A^2}{4})})$ can be obtained as follows. First, we need to introduce a new real stationary point $\mu(\xi)$ which must be a zero of the new differential $d\hat{g}$. On the other hand we have to preserve the asymptotic behavior of the g-function for large λ. To do so we must change the denominator of the differential $d\hat{g}$. Thus the new differential takes the form:

$$d\hat{g}(\lambda,\xi) = 4\frac{(\lambda-\mu(\xi))(\lambda-\mu_-(\xi))(\lambda-\mu_+(\xi))}{\sqrt{(\lambda-E)(\lambda-\bar{E})(\lambda-d(\xi))(\lambda-\bar{d}(\xi))}}d\lambda, \tag{4.36}$$

where $\mu(\xi), \mu_\pm(\xi)$, and $d(\xi), \bar{d}(\xi)$ are to be determined.

If $\mu = d = \bar{d}$, then the new differential coincides with the previous one, that is $dg = d\hat{g}$, which is expected to hold for the value ξ_0 of ξ limiting the two adjacent asymptotic regions.

Now we consider $d\hat{g}$ as an Abelian differential of the second kind with poles at $\infty_\pm$ on the Riemann-Hilbert surface of

$$\omega(\lambda) = \sqrt{(\lambda-E)(\lambda-\bar{E})(\lambda-d(\xi))(\lambda-\bar{d}(\xi))},$$

with

$$E = B+iD, \quad d(\xi) = d_1(\xi)+id_2(\xi)$$

The branch of the square root is fixed by the asymptotics on the upper sheet:

$$\omega(\lambda) = \lambda^2 + O(\lambda), \qquad \lambda \to \infty_+.$$

We choose on this Riemann surface a basis $\{a,b\}$ of cycles as follows. The b-cycle is a closed clock-wise oriented simple loop around the arc $\gamma_{E,d}$ joining E and d. The a-cycle starts on the upper sheet from the

left side of the cut $\gamma_{E,d}$, goes to the left side of the cut $\gamma_{\bar{d},\bar{E}}$, proceeds to the lower sheet, and then returns to the starting point.

We can also write the Abelian differential $d\hat{g}(\lambda)$ in the form:

$$d\hat{g}(\lambda) = 4\frac{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0}{\omega(\lambda)}d\lambda, \tag{4.37}$$

and normalize it so that its $a-$period vanishes. This determines c_0:

$$c_0 = -\frac{\int_{\bar{d}}^{d}(\lambda^3 + c_2\lambda^2 + c_1\lambda)\frac{d\lambda}{\omega(\lambda)}}{\int_{\bar{d}}^{d}\frac{d\lambda}{\omega(\lambda)}} \in \mathbb{R}.$$

We also require that $\hat{g}(\lambda)$ has the same large-λ behavior as the original phase function $\theta(\lambda)$:

$$\hat{g}(\lambda) = 2\lambda^2 t + 4\lambda x + O(1), \qquad \lambda \to \infty_+.$$

This condition implies

$$c_1 = (B-\xi)d_1 - B\xi + \frac{1}{2}(d_2^2 + D^2),$$

$$c_2 = \xi - B - d_1,$$

Define $\hat{g}(\lambda)$ as the sum of two Abelian integrals:

$$\hat{g}(\lambda,\xi) = 2(\int_E^\lambda + \int_{\bar{E}}^\lambda)\frac{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0}{\omega(\lambda)}d\lambda. \tag{4.38}$$

Then it evidently has real $b-$period

$$B_{\hat{g}} = 2(\int_E^d + \int_{\bar{E}}^{\bar{d}})\frac{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0}{\omega(\lambda)}d\lambda. \tag{4.39}$$

Now notice that $\hat{g}(\lambda)$ can be written as a single Abelian integral

$$\hat{g}(\lambda) = 4\int_E^k \frac{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0}{\omega(\lambda)}d\lambda$$

and indeed

$$B_{\hat{g}} = \int_b d\hat{g}.$$

The large-λ asymptotics of $\hat{g}(\lambda,\xi)$ can now be specified as

$$\hat{g}(\lambda,\xi) = 2\lambda^2 t + 4\xi\lambda t + \hat{g}(\infty,\xi) + O(\lambda^{-1}).$$

where

$$\hat{g}(\infty,\xi)=t(2(\int_{E}^{\infty}+\int_{\bar{E}}^{\infty})[\frac{\lambda^3+c_2\lambda^2+c_1\lambda+c_0}{\omega(\lambda)}-(\lambda+\xi)]d\lambda+2D^2-2B^2-4B\xi) \tag{4.40}$$

is a real function of ξ.

Remark 4.4. *For $\xi=-B$, if we set $\mu(-B)=d_1(-B)=B$ and $d_2(-B)=D$, that is, $d(-B)=E$ and $\bar{d}(-B)=\bar{E}$, then $\hat{g}(\lambda,-B)$ coincide(up to a constant) with $\theta(\lambda,-B)$:*

$$\hat{g}(\lambda,-B)=\theta(\lambda,-B)+2|E|^2.$$

which provides matching at the interface with the Zakharov-Manakov region.

In order to define $\mu,\mu_\pm$ and d as functions of ξ, let us compare the forms (4.36) and (4.37) of the differential $d\hat{g}$. This gives ($\mu_\pm=\mu_1\pm i\mu_2$) :

$$\begin{aligned}
&\mu+2\mu_1-d_1=B-\xi,\\
&2\mu\mu_1+\mu_1^2+\mu_2^2+(\xi-B)d_1-\tfrac{1}{2}d_2^2=\tfrac{1}{2}D^2-B\xi,\\
&\mu(\mu_1^2+\mu_2^2)=-c_0(\xi,d_1,d_2).
\end{aligned}$$

The local expansion of $\hat{g}(\lambda)$ at $\lambda=d$ is of the form

$$\hat{g}(\lambda)=B_{\hat{g}}+g_1(\lambda-d)^{1/2}+g_2(\lambda-d)^{3/2}+\cdots,$$

where $B_{\hat{g}}$ is real. The signature table for $\mathrm{Im}\hat{g}(\lambda)$ must have three branches of the curve $\mathrm{Im}\hat{g}(\lambda)=0$ going out from the point d, see Figure 6. Indeed:

- Since $\hat{g}(E)=0$, one branch should connect d with E.
- There should exist a branch separating the basins of $+$ and $-$ near the real axis.
- Since $\hat{g}(\lambda)$ behaves like $\theta(\lambda)$ for large λ, there should be an infinite branch going to infinity along the asymptotic line $\mathrm{Re}\lambda=-\xi$.

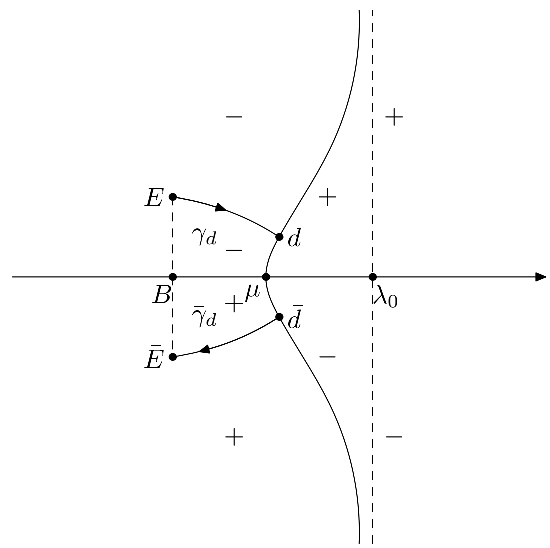

FIGURE 6. The curves of $\mathrm{Im}\hat{g}(\lambda) = 0$ for $-4t(B + \sqrt{2A^2(B + \frac{A^2}{4})}) < x < -4tB$.

Therefore, we arrive at the requirement $g_1 = 0$, that is

$$(\lambda - d)^{1/2}\hat{g}'(\lambda)|_{\lambda=d} = 4\frac{(d-\mu(\xi))(d-\mu_-(\xi))(d-\mu_+(\xi))}{\sqrt{(\lambda-E)(\lambda-\bar{E})(d-\bar{d})}} = 0$$

The fact that μ is real implies that $\mu_+ = d$ and $\mu_- = \bar{d}$, which finally leads to the following ansatz for $d\hat{g}(\lambda)$:

$$d\hat{g}(\lambda) = 4(\lambda - \mu(\xi))\sqrt{\frac{(\lambda - d(\xi))(\lambda - \bar{d}(\xi))}{(\lambda - E)(\lambda - \bar{E})}}d\lambda,$$

where $\mu(\xi), d_1(\xi)$ and $d_2(\xi)$ ($d = d_1 + id_2, d_2 \geq 0$) satisfy the equations:

$$\mu = B - \xi - d_1, \tag{4.41a}$$

$$d_2^2 = D^2 - 2(B-\mu)(B-d_1), \tag{4.41b}$$

$$\int_{B-iD}^{B+iD} \sqrt{\frac{(\lambda - d_1)^2 + d_2^2}{(\lambda - B)^2 + D^2}}(\lambda - \mu)d\lambda = 0. \tag{4.41c}$$

Recall that (4.41a) and (4.41b) follow from the requirement that

$$d\hat{g}(\lambda) = (4\lambda + 4\xi + O(\lambda^{-2}))d\lambda, \qquad \text{as } \lambda \to \infty.$$

while (4.41c) is the normalization condition $\int_{\bar{E}}^{E} d\hat{g}(\lambda) = 0$.

Substituting (4.41a) and (4.41b) into (4.41c) yields an equation relating implicitly d_1 and ξ. In terms of the variables u and v, where

$$u = \frac{B - d_1}{D}, \qquad v = \frac{\xi + B}{2D}.$$

this equation reads

$$\mathcal{F}(u,v) = \int_{-1}^{1} \sqrt{\frac{(i\tau+1)^2 + 1 - 4uv + 2u^2}{1-\tau^2}}(i\tau + 2v - u)d\tau = 0. \quad (4.42)$$

which is considered for $0 \leq v \leq \frac{\sqrt{2}}{2}$ and $u \geq 0$. It is easy to check that $\mathcal{F}(0,v) = 4v$(and thus $\mathcal{F}(0,v) > 0$ for $v > 0$), $\mathcal{F}(+\infty, v) < 0, \mathcal{F}(0,0) = \mathcal{F}(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 0$ and $\mathcal{F}_u(u,v) < 0$ for $(u,v) \neq (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Therefore, (4.42) determines a unique function $u = u(v), v \in [0, \frac{\sqrt{2}}{2}]$ such that $u(0) = 0$ and $u(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}$. Consequently, we have that the system (4.41) determines uniquely $d_1(\xi), d_2(\xi)$ and $\mu(\xi)$, such that $d_1(-B - \sqrt{2}D) = B + \sqrt{2}D$ and $d_1(-B) = B$.

We have now specified a $g-$function $\hat{g}(\lambda)$ whose signature table is as in Figure 8. Hence, we can begin deforming the basic Riemann-Hilbert problem.

4.3.2. *The first deformation.* We deform the part $\gamma \cup \bar{\gamma}$ of the contour of the basic Riemann-Hilbert problem into a contour $\gamma_{E,\bar{E}}$ connecting E and $\bar{E}$ in such a way that it contains:

(i) Two arcs γ_d and $\bar{\gamma}_d$ connecting, respectively, E with d and $\bar{d}$ and $\bar{E}$, and where $\mathrm{Im}\hat{g}(\lambda) = 0$;
(ii) An arc γ_μ connecting d and $\bar{d}$, passing through μ, and along which $\mathrm{Im}\hat{g}(\lambda) < 0$ for $\mathrm{Im}\lambda < 0$ and $\mathrm{Im}\hat{g}(\lambda) > 0$ for $\mathrm{Im}\lambda > 0$.

Supplying $\gamma_{E,\bar{E}} = \gamma_\mu \cup \gamma_d \cup \bar{\gamma}_d$ with the orientation as going from E to $\bar{E}$, we fix the branch of $\hat{g}(\lambda)$ as having a jump across $\gamma_{E,\bar{E}}$:

$$\begin{aligned} &\hat{g}(\lambda)_+ + \hat{g}(\lambda)_- = 0, && \lambda \in \gamma_d \cup \bar{\gamma}_d; \\ &\hat{g}(\lambda)_+ - \hat{g}(\lambda)_- = B_{\hat{g}}, && \lambda \in \gamma_\mu, \\ &\text{with } \mathrm{Im}B_{\hat{g}} = 0 \end{aligned}$$

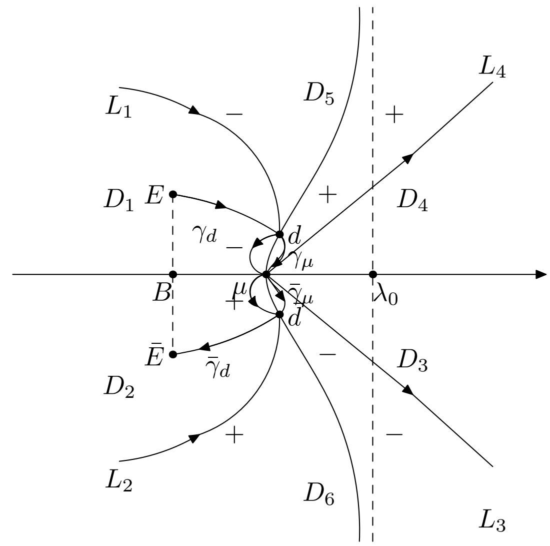

FIGURE 7. The contour $\Sigma^{(3)} = L_1 \cup L_2 \cup L_3 \cup L_4 \cup \gamma_d \cup \bar{\gamma}_d \cup \gamma_\mu \cup \bar{\gamma}_\mu$ for $-4t(B + \sqrt{2A^2(B + \frac{A^2}{4})}) < x < -4tB$.

4.3.3. *The second transformation.* The further series of transformations

$$N(x,t,\lambda) \rightsquigarrow N^{(1)}(x,t,\lambda) \rightsquigarrow N^{(2)}(x,t,\lambda) \rightsquigarrow N^{(3)}(x,t,\lambda)$$

is similar to that for the plane wave region but

(i) with $g(\lambda)$ replaced by $\hat{g}(\lambda)$,
(ii) with μ, which is the real stationary point of $\hat{g}(\lambda)$ instead of μ_+,
(iii) with the partition into domains with boundaries L as shown in Figure 7.

The jump matrix $J_N^{(3)}(x,t,\lambda)$ is as follows:

- For $\lambda \in L_j$ at a fixed positive distance from the stationary point $\lambda = \mu(\xi)$,

$$J_N^{(3)}(x,t,\lambda) = \mathbb{I} + O(e^{-\varepsilon t}) \text{ as } t \to +\infty.$$

- For $\lambda \in \gamma_\mu$ we have

$$J_N^{(3)}(x,t,\lambda) = \begin{cases} \begin{pmatrix} e^{-itB_{\hat{g}}} & 0 \\ \lambda f(\lambda)\delta^{-2}(\lambda)e^{it(\hat{g}_+(\lambda)+\hat{g}_-(\lambda))} & e^{itB_{\hat{g}}} \end{pmatrix}, & \mathrm{Im}\lambda > 0, \\ \begin{pmatrix} e^{-itB_{\hat{g}}} & f(\lambda)\delta^{2}(\lambda)e^{-it(\hat{g}_+(\lambda)+\hat{g}_-(\lambda))} \\ 0 & e^{itB_{\hat{g}}} \end{pmatrix}, & \mathrm{Im}\lambda < 0, \end{cases} \tag{4.43}$$

Thus, away from d,μ and $\bar{d}$ and as $t \to +\infty$, $J_N^{(3)}(x,t,\lambda)$ is close to a diagonal matrix:

$$J_N^{(3)}(x,t,\lambda) = \begin{pmatrix} e^{-itB_{\hat{g}}} & 0 \\ 0 & e^{itB_{\hat{g}}} \end{pmatrix} + O(e^{-\varepsilon t}), \qquad t \to +\infty. \tag{4.44}$$

- For $\lambda \in \gamma_d \cup \bar{\gamma}_d$, similarly to the plane wave region, $J_N^{(3)}(x,t,\lambda)$ reduces to

$$J_N^{(3)}(x,t,\lambda) = \begin{cases} \begin{pmatrix} 0 & -f^{-1}(\lambda)\delta^{2}(\lambda) \\ \lambda f(\lambda)\delta^{-2}(\lambda) & 0 \end{pmatrix}, & \lambda \in \gamma_d, \\ \begin{pmatrix} 0 & f(\lambda)\delta^{2}(\lambda) \\ -\lambda f^{-1}(\lambda)\delta^{-2}(\lambda) & 0 \end{pmatrix}, & \lambda \in \bar{\gamma}_d, \end{cases} \tag{4.45}$$

In order to arrive at a Riemann-Hilbert problem with a jump matrix independent of λ, we proceed as in the plane wave region.

Scalar Riemann-Hilbert problem. We are looking for a scalar function $F(\lambda)$ analytic in $\mathbb{C}\backslash\gamma_d \cup \bar{\gamma}_d$ such that

$$F_-(\lambda)F_+(\lambda) = h(\lambda)\sqrt{\lambda}\delta^{-2}(\lambda), \qquad \lambda \in \gamma_d \cup \bar{\gamma}_d, \tag{4.46}$$

where

$$h(\lambda) = \begin{cases} -i\sqrt{\lambda}f(\lambda), & \lambda \in \gamma_g, \\ i\sqrt{\lambda}^{-1}f^{-1}(k), & \lambda \in \bar{\gamma}_g. \end{cases} \tag{4.47}$$

After solving this scalar problem, $J_N^{(3)}(x,t,\lambda)$ can be factorized as in (4.25). This factorization allows absorbing the diagonal factors into a new Riemann-Hilbert problem with constant jump matrix on $\gamma_d \cup \bar{\gamma}_d$.

However, an important difference with the plane wave region is that now the jump conditions (4.46) for $F(\lambda)$ are specified on two disjoint

arcs. This implies that in order to arrive at a jump condition in additive form, we are led to use

$$\omega(\lambda)=\sqrt{(\lambda-E)(\lambda-\bar{E})(\lambda-d(\xi))(\lambda-\bar{d}(\xi))}$$

Indeed, (4.46) can be rewritten as

$$[\frac{\log F(\lambda)}{\omega(\lambda)}]_+-[\frac{\log F(\lambda)}{\omega(\lambda)}]_-=\frac{\log h(\lambda)}{\omega_+(\lambda)},\quad \lambda\in\gamma_d\cup\bar{\gamma}_d, \tag{4.48}$$

and thus for $F(\lambda)$, we have

$$F(\lambda)=\exp\{\frac{\omega(\lambda)}{2\pi i}\int_{\gamma_d\cup\bar{\gamma}_d}\frac{\log h(s)}{\omega_+(s)}\frac{ds}{s-\lambda}\} \tag{4.49}$$

But now $F(\lambda)$ has an essential singularity at infinity:

$$F(\lambda)=F_\infty e^{i\Delta\lambda}(1+O(\lambda^{-1})),\qquad \lambda\to\infty.$$

where

$$\Delta=\Delta(\xi)=\frac{1}{2\pi}\int_{\gamma_d\cup\bar{\gamma}_d}\frac{\log h(\lambda)}{\omega_+(\lambda)}d\lambda. \tag{4.50}$$

and

$$F_\infty(\xi)=\exp\{\frac{i}{2\pi}\int_{\gamma_d\cup\bar{\gamma}_d}(s-e_1)\frac{\log h(s)}{\omega_+(s)}ds\}$$

with

$$e_1=\frac{E+\bar{E}+d+\bar{d}}{2}. \tag{4.51}$$

To account for this singularity, let us introduce the normalized, that is, its $a-$period vanishes, Abelian integral $w(\lambda)$ of the second kind with simple poles at $\infty_\pm$:

$$w(\lambda)=\int_E^\lambda\frac{z^2-e_1z+e_0}{\omega_(z)}dz,$$

where e_1 is the same as in (4.51) and e_0 is determined by the condition $\int_a dw(\lambda)=0$:

$$e_0=-\frac{\int_d^{\bar{d}}(z^2-e_1z+e_0)\frac{dz}{\omega_(z)}}{\int_d^{\bar{d}}\frac{dz}{\omega_(z)}}.$$

The large-λ expansion of $w(\lambda)$ is of the form

$$w(\lambda)=\lambda+w_\infty(\xi)+O(\lambda^{-1}),\qquad \lambda\to\infty,$$

where

$$\begin{aligned} w_\infty &= \int_E^\infty [\frac{z^2 - e_1 z + e_0}{\omega_(z)} - 1] dz - E \\ &= \frac{1}{2}(\int_E^\infty + \int_{\bar{E}}^\infty)[\frac{z^2 - e_1 z + e_0}{\omega_(z)} - 1] dz - B \end{aligned} \tag{4.52}$$

The jump conditions for $w(\lambda)$ are as follows:

$$\begin{aligned} w_+(\lambda) + w_-(\lambda) &= 0, \qquad \lambda \in \gamma_d \cup \bar{\gamma}_d, \\ w_+(\lambda) - w_-(\lambda) &= B_w, \qquad \lambda \in \gamma_\mu. \end{aligned}$$

Here B_w is the $b-$period of $w(\lambda)$:

$$B_w = \int_b dw = 2\int_E^d \frac{z^2 - e_1 z + e_0}{\omega_(z)} dz = (\int_E^d + \int_{\bar{E}}^{\bar{d}}) \frac{z^2 - e_1 z + e_0}{\omega_(z)} dz \in \mathbb{R}. \tag{4.53}$$

Now introduce

$$\hat{F}(\lambda) = F(\lambda) e^{-i\Delta w(\lambda)}, \tag{4.54}$$

This new function is clearly bounded at $\lambda = \infty$:

$$\hat{F}(\infty, \xi) = e^{i\hat{\phi}(\xi)}. \tag{4.55}$$

with

$$\hat{\phi}(\xi) = \frac{1}{2\pi} \int_{\gamma_d \cup \bar{\gamma}_d} (s - e_1) \log [h(s)\delta^{-2}(s, \xi)] \frac{ds}{\omega_+(s)} - \Delta(\xi) w_\infty(\xi).$$

Also, $\hat{F}(\lambda)$ has the same jumps as $F(\lambda)$ across γ_d and $\bar{\gamma}_d$. On the other hand, the price for introducing the exponential factor in (4.54) is that $\hat{F}(\lambda)$ has a jump across γ_μ:

$$\frac{\hat{F}_+(\lambda)}{\hat{F}_-(\lambda)} = e^{-i\Delta B_w}, \qquad \lambda \in \gamma_\mu.$$

Now we can absorb $\hat{F}(\lambda)$ into the Riemann-Hilbert problem for $N^{(4)}(x, t, \lambda)$:

$$N^{(4)}(x, t, \lambda) = \hat{F}^{\sigma_3}(\infty) N^{(3)}(x, t, \lambda) \hat{F}^{-\sigma_3}(\lambda),$$

which leads to the jump conditions

$$N_+^{(4)}(x, t, \lambda) = N_-^{(4)}(x, t, \lambda) J_N^{(4)}(x, t, \lambda),$$

where

$$J_N^{(4)}(x,t,\lambda) = \begin{cases} J_N^{mod} + O(e^{-\varepsilon t}), & \lambda \in \gamma_d \cup \bar{\gamma}_d \cup \gamma_\mu, \\ \mathbb{I} + O(e^{-\varepsilon t}), & \lambda \in L \cup \bar{L}. \end{cases}$$

with

$$J_N^{(mod)} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \lambda \in \gamma_d \cup \bar{\gamma}_d, \\ \begin{pmatrix} e^{-itB_{\hat{g}} - i\Delta B_w} & 0 \\ 0 & e^{itB_{\hat{g}} + i\Delta B_w} \end{pmatrix}, & \lambda \in \gamma_\mu, \end{cases} \tag{4.56}$$

4.3.4. *The model problem.* Thus, we arrive at the model Riemann-Hilbert problem:

$$N_+^{mod}(x,t,\lambda) = N_-^{mod}(x,t,\lambda) J_N^{mod}(x,t,\lambda), \qquad \lambda \in \gamma_d \cup \bar{\gamma}_d \cup \gamma_\mu, \tag{4.57a}$$

$$N^{mod}(x,t,\lambda) = \mathbb{I} + O(\frac{1}{\lambda}), \qquad \lambda \to \infty. \tag{4.57b}$$

The solution of this model Riemann-Hilbert problem approximates $N^{(4)}(x,t,\lambda)$:

$$N^{(4)}(x,t,\lambda) = (\mathbb{I} + O(t^{-\frac{1}{2}}))N^{mod}(x,t,\lambda), \tag{4.58}$$

The model problem (4.57) can be solved in terms of elliptic theta functions. Let

$$U(\lambda) = \frac{1}{c}\int_E^\lambda \frac{dz}{\omega(z)}$$

be the normalized Abelian integral, that is

$$c = 2\int_{\bar{d}}^d \frac{dz}{\omega(z)}$$

Then, define

$$\tau = \tau(\xi) = \frac{2}{c}\int_E^d \frac{dz}{\omega(z)} \tag{4.59}$$

with $\mathrm{Im}\tau > 0$. Furthermore, the following relations are valid:

$$\begin{aligned} U_+(\lambda) + U_-(\lambda) &= 0, && \lambda \in \gamma_d, \\ U_+(\lambda) + U_-(\lambda) &= -1, && \lambda \in \bar{\gamma}_d, \\ U_+(\lambda) - U_-(\lambda) &= \tau, && \lambda \in \gamma_\mu, \end{aligned} \tag{4.60}$$

Next, define

$$\nu(\lambda)=(\frac{(\lambda-E)(\lambda-d)}{(\lambda-\bar{E})(\lambda-\bar{d})})^{\frac{1}{4}},$$

where the branch is fixed by specifying the branch cut $\gamma_{E,\bar{E}}$ and the behavior as $\lambda\to\infty$;

$$\nu(\lambda)=1+\frac{D+d_2}{2i\lambda}+O(\lambda^{-2}),\qquad \lambda\to\infty.$$

Along the cut,we have

$$\nu_+(\lambda)=\begin{cases} -i\nu_-(\lambda), & \lambda\in\gamma_d\cup\bar{\gamma}_d,\\ -\nu_-(\lambda), & \lambda\in\gamma_\mu.\end{cases}$$

Finally, introduce the theta function

$$\theta_3(z)=\sum_{m\in\mathbb{Z}}e^{\pi i\tau m^2+2\pi imz},$$

and define the 2×2 matrix-value function $\Theta(\lambda)=\Theta(t,\xi,\lambda)$ with entries:

$$\begin{aligned}
\Theta_{11}(\lambda)&=\tfrac{1}{2}[\nu(\lambda)+\tfrac{1}{\nu(\lambda)}]\tfrac{\theta_3[U(\lambda)-U_0-\frac{1}{2}-\frac{B_{\hat{g}}t}{2\pi}-\frac{B_w\Delta}{2\pi}]}{\theta_3[U(\lambda)-U_0]},\\
\Theta_{12}(\lambda)&=\tfrac{1}{2}[\nu(\lambda)-\tfrac{1}{\nu(\lambda)}]\tfrac{\theta_3[U(\lambda)+U_0+\frac{1}{2}+\frac{B_{\hat{g}}t}{2\pi}+\frac{B_w\Delta}{2\pi}]}{\theta_3[U(\lambda)+U_0]},\\
\Theta_{21}(\lambda)&=\tfrac{1}{2}[\nu(\lambda)-\tfrac{1}{\nu(\lambda)}]\tfrac{\theta_3[U(\lambda)+U_0-\frac{1}{2}-\frac{B_{\hat{g}}t}{2\pi}-\frac{B_w\Delta}{2\pi}]}{\theta_3[U(\lambda)+U_0]},\\
\Theta_{22}(\lambda)&=\tfrac{1}{2}[\nu(\lambda)+\tfrac{1}{\nu(\lambda)}]\tfrac{\theta_3[U(\lambda)-U_0+\frac{1}{2}+\frac{B_{\hat{g}}t}{2\pi}+\frac{B_w\Delta}{2\pi}]}{\theta_3[U(\lambda)-U_0]},
\end{aligned}$$

where U_0 is to be chosen so that the unique zero of $\theta_3(U(\lambda)-U_0)$, as a function on the Riemann surface, lying on the first sheet is compensated by the zero of $\nu(\lambda)+\frac{1}{\nu(\lambda)}$ where $\theta_3(U(\lambda)+U_0)$ has no zero on this sheet. Setting

$$U_0=U(E_0)+\frac{1}{2}+\frac{\tau}{2},$$

where

$$E_0=\frac{Ed-\bar{E}\bar{d}}{E-\bar{E}+d-\bar{d}}$$

satisfies this requirement, and thus $\Theta(\lambda)$ can be viewed as a function analytic in $\mathbb{C}\backslash\gamma_{E,\bar{E}}$. On the other hand, due to the properties of theta function:

$$\theta_3(-z)=\theta_3(z),\quad \theta_3(z+1)=\theta_3(z),\quad \theta_3(z\pm\tau)=e^{-\pi i\tau\mp 2\pi iz}\theta_3(z)$$

$\Theta(\lambda)$ satisfies the jump conditions (4.57a)-(4.56) of the model Riemann-Hilbert problem. Taking into account the normalization condition (4.57b), the solution of the model Riemann-Hilbert problem is given by

$$N^{mod}(x,t,\lambda) = \Theta^{-1}(t,\xi,\infty)\Theta(t,\xi,\lambda).$$

4.3.5. *Back to the original problem.* Now, following the sequence of equations of type (4.34) (with g and F replaced, respectively, by $\hat{g}$ and $\hat{F}$) and taking into account the equations $\hat{g}$ and $\hat{F}$, and the explicit formula for $n_{12}^{mod}(x,t,\lambda)$

$$2in_{12}^{mod}(x,t,\lambda) = [D+d_2]\frac{\theta_3[\frac{B_{\hat{g}}t}{2\pi} + \frac{B_w\Delta}{2\pi} + U_0 + \frac{1}{2} + U(\infty)]}{\theta_3[\frac{B_{\hat{g}}t}{2\pi} + \frac{B_w\Delta}{2\pi} + U_0 + \frac{1}{2} - U(\infty)]}\frac{\theta_3[U_0 - U(\infty)]}{\theta_3[U_0 + U(\infty)]}$$

and $\hat{F}^{-2}(\infty) = e^{-2i\hat{\phi}(\xi)}$, we obtain the asymptotics in the region $-4t(B+\sqrt{2A^2(B+\frac{A^2}{4})}) < x < -4tB$.

Theorem 4.5. (**Elliptic wave region**) *In the region* $-4t(B+\sqrt{2A^2(B+\frac{A^2}{4})}) < x < -4tB$,*the asymptotics, as* $t \to +\infty$, *of the solution* $q(x,t)$ *of the initial value problem (1.6) takes the form of a modulated elliptic wave:*

$$q(x,t) = [D+\mathrm{Im}d(\xi)]\frac{\theta_3[\frac{B_{\hat{g}}t}{2\pi} + \frac{B_w\Delta}{2\pi} + V_+(\xi)]}{\theta_3[\frac{B_{\hat{g}}t}{2\pi} + \frac{B_w\Delta}{2\pi} + V_-(\xi)]}\frac{\theta_3[V_-(\xi) - \frac{1}{2}]}{\theta_3[V_+(\xi) - \frac{1}{2}]} + O(t^{-\frac{1}{2}}), t \to +\infty. \tag{4.61}$$

Here $B_{\hat{g}}, B_w$ *and* Δ *are functions of the variable* $\xi = \frac{x}{4t}$ *defined, respectively, by (4.39), (4.53) and (4.50), and* $V_\pm(\xi) = U_0 + \frac{1}{2} \pm U(\infty)$. *Furthermore,*

$$\theta_3(z) = \sum_{z\in\mathbb{Z}} e^{\pi i\tau m^2 + 2\pi imz}$$

is the theta function of invariant $\tau = \tau(\xi)$ *defined in (4.59),*

$$\hat{g}(\infty,\xi) = t(2(\int_E^\infty + \int_{\bar{E}}^\infty)[(z-\mu(\xi))\sqrt{\frac{(z-d(\xi))(z-\bar{d}(\xi))}{(z-E)(z-\bar{E})}} - (z+\xi)]dz + 2D^2 - 2B^2 - 4B\xi)$$

and the phase shift $\phi(\xi)$ *is given by*

$$\phi(\xi) = \frac{1}{2\pi}\int_{\gamma_d\cup\gamma_{\bar{d}}} \frac{[s - e_1(\xi) - \omega_\infty(\xi)]\log[h(s)\sqrt{s}\delta^{-2}(s,\xi)]}{[(s-E)(s-\bar{E})(s-d(\xi))(s-\bar{d}(\xi))]^{1/2}}ds$$

where

$$h(\lambda) = \begin{cases} a_+^{-1}(\lambda)a_-^{-1}(\lambda), & \lambda \in \gamma_d \\ a_+(\lambda)a_-(\lambda), & \lambda \in \gamma_{\bar{d}} \end{cases}$$
$$\delta(\lambda, \xi) = \exp\{\tfrac{1}{2\pi i}\int_{-\infty}^{\mu(\xi)} \tfrac{\log(1+\lambda\rho^2(\lambda))}{s-\lambda} ds\}.$$

and $e_1(\xi), \omega_\infty$ *and* $\mu(\xi)$ *are defined, respectively, by (4.51), (4.52) and (4.41).*

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